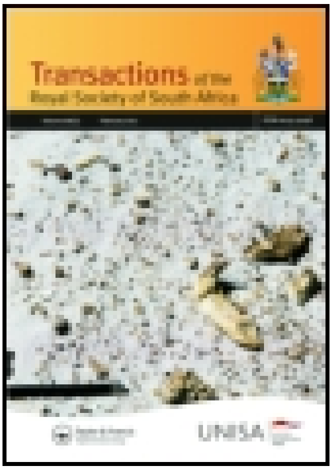

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NOTE ON DOUBLE ALTERNANTS

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By Thomas Muir, LL.D., F.R.S.

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1. The first form of alternant to which it is desired to direct attention is the particular case of

$$\left|(\alpha_1+\beta_1)^p(\alpha_2+\beta_2)^p\ldots(\alpha_n+\beta_n)^p\right|, \qquad \text{or } \mathrm{D}_{n\,;\,p} \text{ say,}$$

where $p=n$, the case where $p=n-1$ having been already dealt with by Zehfuss (*Zeitschrift f. Math. u. Phys.* iv. pp. 233–236). The problem is, of course, to find the quotient resulting from dividing $\mathrm{D}_{n\,;\,n}$ by the difference-product of the α's and the difference-product of the β's—that is to say, the quotient

$$\mathrm{D}_{n\,;\,n}\div\zeta^{\frac{1}{2}}(\alpha_1,\,\alpha_2,\,\ldots,\,\alpha_n)\,.\,\zeta^{\frac{1}{2}}(\beta_1,\,\beta_2,\,\ldots,\,\beta_n),$$

or say

$$\mathrm{D}_{n\,;\,n}\div\zeta_1^{\frac{1}{2}}\zeta_2^{\frac{1}{2}}.$$

2. It is readily seen that by row-by-row multiplication we have

$$\begin{vmatrix} \alpha_1^3 & 3\alpha_1^2 & 3\alpha_1 & 1 \\ \alpha_2^3 & 3\alpha_2^2 & 3\alpha_2 & 1 \\ \alpha_3^3 & 3\alpha_3^2 & 3\alpha_3 & 1 \\ -\beta_1\beta_2\beta_3 & \Sigma\beta_1\beta_2 & -\Sigma\beta_1 & 1 \end{vmatrix} \cdot \begin{vmatrix} 1 & \beta_1 & \beta_1^2 & \beta_1^3 \\ 1 & \beta_2 & \beta_2^2 & \beta_2^3 \\ 1 & \beta_3 & \beta_3^2 & \beta_3^3 \\ 1 & x & x^2 & x^3 \end{vmatrix}$$

$$= \begin{vmatrix} (\alpha_1+\beta_1)^3 & (\alpha_1+\beta_2)^3 & (\alpha_1+\beta_3)^3 & (\alpha_1+x)^3 \\ (\alpha_2+\beta_1)^3 & (\alpha_2+\beta_2)^3 & (\alpha_2+\beta_3)^3 & (\alpha_2+x)^3 \\ (\alpha_3+\beta_1)^3 & (\alpha_3+\beta_2)^3 & (\alpha_3+\beta_3)^3 & (\alpha_3+x)^3 \\ \cdot & \cdot & \cdot & (x-\beta_1)(x-\beta_2)(x-\beta_3) \end{vmatrix},$$

and that on dividing both sides of this by $(x-\beta_1)(x-\beta_2)(x-\beta_3)$ there results

$$\mathrm{D}_{3\,;\,3} = \begin{vmatrix} \alpha_1^3 & 3\alpha_1^2 & 3\alpha_1 & 1 \\ \alpha_2^3 & 3\alpha_2^2 & 3\alpha_2 & 1 \\ \alpha_3^3 & 3\alpha_3^2 & 3\alpha_3 & 1 \\ -\beta_1\beta_2\beta_3 & \Sigma\beta_1\beta_2 & -\Sigma\beta_1 & 1 \end{vmatrix} \zeta^{\frac{1}{2}}(\beta_1,\,\beta_2,\,\beta_3).$$

But the four-line determinant here is seen to be divisible by $a_3 - a_2$, $a_3 - a_1$, $a_2 - a_1$; and, these factors being removed in the ordinary way, we have

$$D_{3;3} = \begin{vmatrix} a_1^3 & 3a_1^2 & 3a_1 & 1 \\ a_2^2 + a_2 a_1 + a_1^2 & 3(a_1 + a_2) & 3 & . \\ a_3 + a_2 + a_1 & 3 & . & . \\ -\beta_1\beta_2\beta_3 & \Sigma\beta_1\beta_2 & -\Sigma\beta_1 & 1 \end{vmatrix} \zeta_2^{\frac{1}{2}}\zeta_1^{\frac{1}{2}}.$$

Further simplification is effected by performing the operations

$$\begin{aligned} \text{row}_1 - \quad a_1.\text{row}_2 + \quad & a_1 a_2.\text{row}_3, \\ \text{row}_2 - \quad & (a_1 + a_2).\text{row}_3, \end{aligned}$$

the penultimate and final results being

$$\frac{D_{3;3}}{\zeta_1^{\frac{1}{2}}\zeta_2^{\frac{1}{2}}} = \begin{vmatrix} a_1 a_2 a_3 & . & . & 1 \\ -\Sigma a_1 a_2 & . & 3 & . \\ \Sigma a_1 & 3 & . & . \\ -\beta_1\beta_2\beta_3 & \Sigma\beta_1\beta_2 & -\Sigma\beta_1 & 1 \end{vmatrix},$$

$$= \begin{vmatrix} a_1 a_2 a_3 + \beta_1\beta_2\beta_3 & \Sigma\beta_1\beta_2 & \Sigma\beta_1 \\ \Sigma a_1 a_2 & . & -3 \\ \Sigma a_1 & -3 & . \end{vmatrix}.$$

Proceeding in exactly similar fashion we obtain

$$\frac{D_{4;4}}{\zeta_1^{\frac{1}{2}}\zeta_2^{\frac{1}{2}}} = \begin{vmatrix} a_1 a_2 a_3 a_4 + \beta_1\beta_2\beta_3\beta_4 & \Sigma\beta_1\beta_2\beta_3 & \Sigma\beta_1\beta_2 & \Sigma\beta_1 \\ \Sigma a_1 a_2 a_3 & . & . & -4 \\ \Sigma a_1 a_2 & . & -6 & . \\ \Sigma a_1 & -4 & . & . \end{vmatrix}$$

and so on, generally. (I.)

3. The form of quotient obtained is manifestly invariant (1) to the interchange of any two a's, (2) to the interchange of any two β's, and (3) to the simultaneous interchange of every a with the corresponding β; and this, as we know, is what ought to be.

4. On account of the number of zero elements in the quotient, it is possible to put it in a simple non-determinant form: thus

$$D_{2;2} \div \zeta_1^{\frac{1}{2}}\zeta_2^{\frac{1}{2}} = -2(a_1 a_2 + \beta_1\beta_2) - \Sigma a_1.\Sigma\beta_1,$$
$$D_{3;3} \div \zeta_1^{\frac{1}{2}}\zeta_2^{\frac{1}{2}} = -3.3(a_1 a_2 a_3 + \beta_1\beta_2\beta_3) - 3(\Sigma a_1 a_2.\Sigma\beta_1 + \Sigma a_1.\Sigma\beta_1\beta_2),$$

....

or, still more interestingly as regards the right-hand members,

$$-1.2.1\left\{\alpha_1\alpha_2 + \tfrac{1}{2}\Sigma\alpha_1.\Sigma\beta_1 + \beta_1\beta_2\right\},$$
$$-1.3.3.1\left\{\alpha_1\alpha_2\alpha_3 + \tfrac{1}{3}\Sigma\alpha_1\alpha_2.\Sigma\beta_1 + \tfrac{1}{3}\Sigma\alpha_1.\Sigma\beta_1\beta_2 + \beta_1\beta_2\beta_3\right\},$$
$$+1.4.6.4.1\left\{\alpha_1\alpha_2\alpha_3\alpha_4 + \tfrac{1}{4}\Sigma\alpha_1\alpha_2\alpha_3.\Sigma\beta_1 + \tfrac{1}{6}\Sigma\alpha_1\alpha_2.\Sigma\beta_1\beta_2 + \tfrac{1}{4}\Sigma\alpha_1.\Sigma\beta_1\beta_2\beta_3 + \beta_1\beta_2\beta_3\beta_4\right\},$$

....

Further, we observe that the determinant quotients are unisignant, the common sign being $+$ when n is of the form $4m$ or $4m+1$, and $-$ when of the form $4m+2$ or $4m+3$. (II.)

5. An interesting verificatory proof is reached by taking the asserted result and multiplying it row-wise by $\zeta_2^{\frac{1}{2}}$, and thereafter multiplying the product column-wise by $\zeta_1^{\frac{1}{2}}$. Thus, if we multiply

$$\begin{vmatrix} \alpha_1\alpha_2\alpha_3\alpha_4\alpha_5 + \beta_1\beta_2\beta_3\beta_4\beta_5 & \Sigma\beta_1\beta_2\beta_3\beta_4 & \Sigma\beta_1\beta_2\beta_3 & \Sigma\beta_1\beta_2 & \Sigma\beta_1 \\ \Sigma\alpha_1\alpha_2\alpha_3\alpha_4 & . & . & . & -5 \\ \Sigma\alpha_1\alpha_2\alpha_3 & . & . & -10 & . \\ \Sigma\alpha_1\alpha_2 & . & -10 & . & . \\ \Sigma\alpha_1 & -5 & . & . & . \end{vmatrix}$$

row-wise by $\zeta_2^{\frac{1}{2}}$ in the form

$$\begin{vmatrix} 1 & -\beta_1 & \beta_1^2 & -\beta_1^3 & \beta_1^4 \\ 1 & -\beta_2 & \beta_2^2 & -\beta_2^3 & \beta_2^4 \\ 1 & -\beta_3 & \beta_3^2 & -\beta_3^3 & \beta_3^4 \\ 1 & -\beta_4 & \beta_4^2 & -\beta_4^3 & \beta_4^4 \\ 1 & -\beta_5 & \beta_5^2 & -\beta_5^3 & \beta_5^4 \end{vmatrix}$$

we obtain

$$\begin{vmatrix} \alpha_1\alpha_2\alpha_3\alpha_4\alpha_5 + \beta_1^5 & \cdots & \alpha_1\alpha_2\alpha_3\alpha_4\alpha_5 + \beta_5^5 \\ \Sigma\alpha_1\alpha_2\alpha_3\alpha_4 - 5\beta_1^4 & \cdots & \Sigma\alpha_1\alpha_2\alpha_3\alpha_4 - 5\beta_5^4 \\ \Sigma\alpha_1\alpha_2\alpha_3 + 10\beta_1^3 & \cdots & \Sigma\alpha_1\alpha_2\alpha_3 + 10\beta_5^3 \\ \Sigma\alpha_1\alpha_2 - 10\beta_1^2 & \cdots & \Sigma\alpha_1\alpha_2 - 10\beta_5^2 \\ \Sigma\alpha_1 + 5\beta_1 & \cdots & \Sigma\alpha_1 + 5\beta_5 \end{vmatrix},$$

and the multiplication of this column-wise by $\zeta_1^{\frac{1}{2}}$ in the form

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 & -\alpha_5 \\ \alpha_1^2 & \alpha_2^2 & \alpha_3^2 & \alpha_4^2 & \alpha_5^2 \\ -\alpha_1^3 & -\alpha_2^3 & -\alpha_3^3 & -\alpha_4^3 & -\alpha_5^3 \\ \alpha_1^4 & \alpha_2^4 & \alpha_3^4 & \alpha_4^4 & \alpha_5^4 \end{vmatrix}$$

produces

$$\begin{vmatrix} (\alpha_1+\beta_1)^5 & \dots & (\alpha_1+\beta_5)^5 \\ \dots\dots & \dots\dots & \dots\dots \\ (\alpha_5+\beta_1)^5 & \dots & (\alpha_5+\beta_5)^5 \end{vmatrix}. \qquad \text{(III.)}$$

6. Taking these results along with Scott's of 1879 (*Messenger of Math.* viii. pp. 182–187), we obtain a remarkable identity—possibly the first observed of its kind—giving an expression for a determinant in terms of a permanent, that is to say, a function of one class in terms of another of the directly opposite class. Thus, for the fourth order, we have

$$\begin{vmatrix} \alpha_1\alpha_2\alpha_3\alpha_4+\beta_1\beta_2\beta_3\beta_4 & \Sigma\beta_1\beta_2\beta_3 & \Sigma\beta_1\beta_2 & \Sigma\beta_1 \\ \Sigma\alpha_1\alpha_2\alpha_3 & \cdot & \cdot & -4 \\ \Sigma\alpha_1\alpha_2 & \cdot & -6 & \cdot \\ \Sigma\alpha_1 & -4 & \cdot & \cdot \end{vmatrix} = 4\,{}^{+}\!\begin{vmatrix} \alpha_1+\beta_1 & \dots & \alpha_1+\beta_4 \\ \alpha_2+\beta_1 & \dots & \alpha_2+\beta_4 \\ \alpha_3+\beta_1 & \dots & \alpha_3+\beta_4 \\ \alpha_4+\beta_1 & \dots & \alpha_4+\beta_4 \end{vmatrix}^{+}$$

the connecting factor, which is here 4, being for the nth order

$$(-1)^{\frac{1}{2}n(n-1)} \cdot \frac{n_1 n_2 \dots n_n}{1.2 \dots n}$$

where $n_r = n(n-1)\dots(n-r+1)/1.2\dots r$. (IV.)

7. A direct mode of establishing this identity is something to be desired. All that we can suggest as a substitute is a proof that the two members of it have the same final development. Taking, for example, the permanent of the third order

$${}^{+}\!\begin{vmatrix} \alpha_1+\beta_1 & \alpha_1+\beta_2 & \alpha_1+\beta_3 \\ \alpha_2+\beta_1 & \alpha_2+\beta_2 & \alpha_2+\beta_3 \\ \alpha_3+\beta_1 & \alpha_3+\beta_2 & \alpha_3+\beta_3 \end{vmatrix}^{+}$$

and recalling the fact that the law for the partitionment of determinants with polynomial elements holds also for permanents we obtain

$${}^{+}\!\begin{vmatrix} \alpha_1 & \alpha_1 & \alpha_1 \\ \alpha_2 & \alpha_2 & \alpha_2 \\ \alpha_3 & \alpha_3 & \alpha_3 \end{vmatrix}^{+} + {}^{+}\!\begin{vmatrix} \alpha_1 & \alpha_1 & \beta_3 \\ \alpha_2 & \alpha_2 & \beta_3 \\ \alpha_3 & \alpha_3 & \beta_3 \end{vmatrix}^{+} + \dots\dots + {}^{+}\!\begin{vmatrix} \beta_1 & \beta_2 & \beta_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \beta_1 & \beta_2 & \beta_3 \end{vmatrix}^{+}$$

which (*Educ. Times*, lxv. p. 139)

$$= 6\alpha_1\alpha_2\alpha_3 + 2\Sigma\alpha_1\alpha_2.\beta_3 + 2\Sigma\alpha_1\alpha_2.\beta_2 + 2\Sigma\alpha_1\alpha_2.\beta_1 + 2\Sigma\alpha_1.\beta_2\beta_3 + 2\Sigma\alpha_1.\beta_1\beta_3 + 2\Sigma\alpha_1.\beta_1\beta_2 + 6\beta_1\beta_2\beta_3,$$

$$= 6\alpha_1\alpha_2\alpha_3 + 2\Sigma\alpha_1\alpha_2.\Sigma\beta_1 + 2\Sigma\alpha_1.\Sigma\beta_1\beta_2 + 6\beta_1\beta_2\beta_3,$$

$$= -\tfrac{2}{3}\begin{vmatrix} \alpha_1\alpha_2\alpha_3+\beta_1\beta_2\beta_3 & \Sigma\beta_1\beta_2 & \Sigma\beta_1 \\ \Sigma\alpha_1\alpha_2 & . & -3 \\ \Sigma\alpha_1 & -3 & . \end{vmatrix}. \qquad \text{(V.)}$$

8. Turning now to alternants of the form $D_{n\,;\,n+1}$ let us consider first $D_{2\,;\,3}$. By the multiplication-theorem there is obtained

$$\begin{vmatrix} \alpha_1^3 & 3\alpha_1^2 & 3\alpha_1 & 1 \\ \alpha_2^3 & 3\alpha_2^2 & 3\alpha_2 & 1 \\ \beta_1\beta_2 & -\Sigma\beta_1 & 1 & . \\ . & \beta_1\beta_2 & -\Sigma\beta_1 & 1 \end{vmatrix} \cdot \begin{vmatrix} 1 & \beta_1 & \beta_1^2 & \beta_1^3 \\ 1 & \beta_2 & \beta_2^2 & \beta_2^3 \\ 1 & x & x^2 & x^3 \\ 1 & y & y^2 & y^3 \end{vmatrix}$$

$$= \begin{vmatrix} (\alpha_1+\beta_1)^3 & (\alpha_1+\beta_2)^3 & (\alpha_1+x)^3 & (\alpha_1+y)^3 \\ (\alpha_2+\beta_1)^3 & (\alpha_2+\beta_2)^3 & (\alpha_2+x)^3 & (\alpha_2+y)^3 \\ . & . & (x-\beta_1)(x-\beta_2) & (y-\beta_1)(y-\beta_2) \\ . & . & x(x-\beta_1(x-\beta_2) & y(y-\beta_1)(y-\beta_2) \end{vmatrix},$$

the division of both members of which by

$$(y-x).(y-\beta_2)(y-\beta_1).(x-\beta_2)(x-\beta_1)$$

gives

$$D_{2\,;\,3} = \begin{vmatrix} \alpha_1^3 & 3\alpha_1^2 & 3\alpha_1 & 1 \\ \alpha_2^3 & 3\alpha_2^2 & 3\alpha_2 & 1 \\ \beta_1\beta_2 & -\Sigma\beta_1 & 1 & . \\ . & \beta_1\beta_2 & -\Sigma\beta_1 & 1 \end{vmatrix} \zeta^{\frac{1}{2}}(\beta_1, \beta_2).$$

The four-line determinant here, however, contains the factor $\alpha_2-\alpha_1$, which being removed and a self-evident simplification effected, we have for the remaining determinant

$$\begin{vmatrix} -\alpha_1\alpha_2.\Sigma\alpha_1 & -3\alpha_1\alpha_2 & . & 1 \\ \alpha_2^2+\alpha_2\alpha_1+\alpha_1^2 & 3\Sigma\alpha_1 & 3 & . \\ \beta_1\beta_2 & -\Sigma\beta_1 & 1 & . \\ . & \beta_1\beta_2 & -\Sigma\beta_1 & 1 \end{vmatrix},$$

and by performing on this the operations

$$\text{col}_4 \times \alpha_1\alpha_2, \quad \text{row}_4 \div \alpha_1\alpha_2, \quad \text{row}_2 + \Sigma\alpha_1.\text{row}_1$$

in succession it is transformed into

$$\begin{vmatrix} -\Sigma\alpha_1 & -3 & . & . \\ -\alpha_1\alpha_2 & . & 3 & \Sigma\alpha_1 \\ \beta_1\beta_2 & -\Sigma\beta_1 & 1 & . \\ . & \beta_1\beta_2 & -\Sigma\beta_1 & \alpha_1\alpha_2 \end{vmatrix}.$$

We thus have finally

$$\frac{D_{2;3}}{\zeta_1^{\frac{1}{2}}\zeta_2^{\frac{1}{2}}} = -\begin{vmatrix} \beta_1\beta_2 & -\Sigma\beta_1 & 1 & . \\ -\Sigma\alpha_1 & -3 & . & 1 \\ \alpha_1\alpha_2 & . & -3 & -\Sigma\alpha_1 \\ . & \beta_1\beta_2 & -\Sigma\beta_1 & \alpha_1\alpha_2 \end{vmatrix},$$

where, be it noted, any two elements situated symmetrically with respect to the secondary diagonal do not differ in form, the one being the same symmetric function of the one set of variables as the other is of the other. As a consequence the invariance referred to in §3 holds here also, as it ought.

In exactly similar fashion there is obtained

$$\frac{D_{3;4}}{\zeta_1^{\frac{1}{2}}\zeta_2^{\frac{1}{2}}} = -\begin{vmatrix} -\beta_1\beta_2\beta_3 & \Sigma\beta_1\beta_2 & -\Sigma\beta_1 & 1 & . \\ \Sigma\alpha_1 & 4 & . & . & 1 \\ -\Sigma\alpha_1\alpha_2 & . & 6 & . & -\Sigma\alpha_1 \\ \alpha_1\alpha_2\alpha_3 & . & . & 4 & \Sigma\alpha_1\alpha_2 \\ . & \beta_1\beta_2\beta_3 & -\Sigma\beta_1\beta_2 & \Sigma\beta_1 & -\alpha_1\alpha_2\alpha_3 \end{vmatrix},$$

and so, generally. (VI.)

9. Doubtless a verificatory proof of this result, similar to that of §7, could be devised; and as a matter of fact in the case of $-D_{2;3} \div \zeta_1^{\frac{1}{2}}\zeta_2^{\frac{1}{2}}$ we have only got to multiply row-wise the asserted equivalent by $-(\beta_2-\beta_1)$ in the form

$$\begin{vmatrix} 1 & \beta_1 & \beta_1^2 & . \\ 1 & \beta_2 & \beta_2^2 & . \\ . & . & . & 1 \\ . & . & 1 & . \end{vmatrix}$$

and then multiply column-wise the product so reached by $-\alpha_1\alpha_2(\alpha_2-\alpha_1)$ in the form

$$\begin{vmatrix} \alpha_1^2 & \alpha_2^2 & . \\ \alpha_1 & \alpha_2 & . \\ 1 & 1 & 1 \end{vmatrix}.$$

10. In the third place let us look at alternants of the form $D_{n;\,n+2}$, taking only the simplest case. By proceeding on lines closely analogous to the above, we readily find

$$D_{2;\,4} = \begin{vmatrix} a_1^4 & 4a_1^3 & 6a_1^2 & 4a_1 & 1 \\ a_2^4 & 4a_2^3 & 6a_2^2 & 4a_2 & 1 \\ \beta_1\beta_2 & -\Sigma\beta_1 & 1 & \cdot & \cdot \\ \cdot & \beta_1\beta_2 & -\Sigma\beta_1 & 1 & \cdot \\ \cdot & \cdot & \beta_1\beta_2 & -\Sigma\beta_1 & 1 \end{vmatrix} \zeta_2^{\frac{1}{2}}.$$

and at a farther stage

$$D_{2;\,4} = \begin{vmatrix} -(a_1^2+a_1a_2+a_2^2) & -4\Sigma a_1 & -6 & \cdot & 1 \\ -a_1a_2(a_1+a_2) & -4a_1a_2 & \cdot & 4 & \Sigma a_1 \\ \beta_1\beta_2 & -\Sigma\beta_1 & 1 & \cdot & \cdot \\ \cdot & \beta_1\beta_2 & -\Sigma\beta_1 & 1 & \cdot \\ \cdot & \cdot & \beta_1\beta_2 & -\Sigma\beta_1 & a_1a_2 \end{vmatrix} \zeta_1^{\frac{1}{2}}\zeta_2^{\frac{1}{2}}$$

A column of 0's is then appended and the additional row

$$\Sigma a_1 \quad 4 \quad . \quad . \quad . \quad 1$$

prefixed, with the result that our next simplification brings us to

$$\frac{D_{2;\,4}}{\zeta_1^{\frac{1}{2}}\zeta_2^{\frac{1}{2}}} = -\begin{vmatrix} -\Sigma a_1 & 1 & \cdot & -4 & \cdot & \cdot \\ a_1a_2 & -\Sigma a_1 & 1 & \cdot & -6 & \cdot \\ \cdot & a_1a_2 & -\Sigma a_1 & \cdot & \cdot & -4 \\ \beta_1\beta_2 & \cdot & \cdot & -\Sigma\beta_1 & 1 & \cdot \\ \cdot & \cdot & \cdot & \beta_1\beta_2 & -\Sigma\beta_1 & 1 \\ \cdot & \cdot & a_1a_3 & \cdot & \beta_1\beta_2 & -\Sigma\beta_1 \end{vmatrix}, \qquad \text{(VII.)}$$

a result which again satisfies the tests regarding invariance.

11. A general theorem in regard to $D_{n;\,n+h}$ in agreement with Garbieri's of 1878 (*Giornale di Mat.* xvi. pp. 1–17) is thus foreshadowed. The fact that in the case of some of the resulting determinants the simple symmetric functions of the a's appear in the same element with those of the β's must not be considered an indication to the contrary of this. Indeed it is sufficient to point out that the form of every element in Garbieri's determinant is a bipartite function, and that an integral power of a binomial is a special case of such a function. For example, the bipartite

$$\begin{array}{cccc|c} 1 & x & x^2 & x^3 & \\ \hline a_1 & a_2 & a_3 & a_4 & 1 \\ b_1 & b_2 & b_3 & b_4 & y \\ c_1 & c_2 & c_3 & c_4 & y^2 \\ d_1 & d_2 & d_3 & d_4 & y^3 \end{array}$$

degenerates into $(x+y)^3$ when $a_4, b_3, c_2, d_1 = 1,3,3,1$ and all the other a's, b's, c's, d's vanish.

12. Lastly, a momentary glance may be taken at Cauchy's double alternant, that is to say, the alternant which in our temporary notation is denoted by

$$D_{n;-1}.$$

This when the two sets of variables are identical is axisymmetric and admits of special treatment. Thus, using $D'_{3;-1}$ to stand for

$$\begin{vmatrix} \frac{1}{2a_1} & \frac{1}{a_1+a_2} & \frac{1}{a_1+a_3} \\ \frac{1}{a_2+a_1} & \frac{1}{2a_2} & \frac{1}{a_2+a_3} \\ \frac{1}{a_3+a_1} & \frac{1}{a_3+a_2} & \frac{1}{2a_3} \end{vmatrix}$$

and f_{rs} for $(a_r - a_s) \div (a_r + a_s)$ we have

$$2a_1 . 2a_2 . 2a_3 . D'_{3;-1} = \begin{vmatrix} 1+0 & 1+f_{12} & 1+f_{13} \\ 1-f_{12} & 1+0 & 1+f_{23} \\ 1-f_{13} & 1-f_{23} & 1+0 \end{vmatrix}.$$

But for this latter determinant may be substituted the four-line determinant

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ -1 & . & f_{12} & f_{13} \\ -1 & -f_{12} & . & f_{23} \\ -1 & -f_{13} & -f_{23} & . \end{vmatrix}$$

which on account of the cofactor of the (1,1)th element being equal to 0 may itself be replaced by

$$\begin{vmatrix} . & 1 & 1 & 1 \\ -1 & . & f_{12} & f_{13} \\ -1 & -f_{12} & . & f_{23} \\ -1 & -f_{13} & -f_{23} & . \end{vmatrix}.$$

We thus have finally

$$D'_{3;-1} = \begin{vmatrix} 1 & 1 & 1 \\ & f_{12} & f_{13} \\ & & f_{23} \end{vmatrix}^2 \div 2^3 a_1 a_2 a_3 \qquad \text{(VIII.1)}$$

$$= \left\{ \frac{a_2-a_3}{a_2+a_3} - \frac{a_1-a_3}{a_1+a_3} + \frac{a_1-a_2}{a_1+a_2} \right\}^2 \div 2^3 a_1 a_2 a_3.$$

There is a difference of form in the result when the given determinant is of even order. Thus, in the case of the fourth order, while we come as before to the equation

$$2^4 . \alpha_1\alpha_2\alpha_3\alpha_4 . D'_{4;-1} = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & . & f_{12} & f_{13} & f_{14} \\ -1 & -f_{12} & . & f_{23} & f_{24} \\ -1 & -f_{13} & -f_{23} & . & f_{34} \\ -1 & -f_{14} & -f_{24} & f_{34} & . \end{vmatrix},$$

and partition the determinant on the right into the sum of a five-line and a four-line zero-axial determinant, it is not the latter but the former that vanishes, giving

$$D'_{4;-1} = \begin{vmatrix} f_{12} & f_{13} & f_{14} \\ & f_{23} & f_{24} \\ & & f_{34} \end{vmatrix}^2 \div 2^4{}_1\alpha_2\alpha_3\alpha_4. \qquad \text{(VIII.2)}$$

13. Underlying these results we have evidently the general theorem that *If all the elements of a zero-axial skew determinant be increased by* 1, *the resulting determinant is an exact square, whatever the order may be;* the reason being that where the order is even the value of the determinant is unaltered by the change, and where the order is odd the new determinant is expressible as a zero-axial skew determinant of the next higher order. (IX.)